

# Joyce invariants for K3 surfaces and mock theta functions

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## Abstract

We will discuss Joyce invariants of stability conditions for K3 surfaces and mock theta functions.

## 1 Introduction

Bridgeland introduced the notion of stability conditions on triangulated categories [Br07], this notion extends standard stabilities such as Gieseker stabilities on the abelian category of coherent sheaves of a variety  $X$ , denoted by  $\text{Coh } X$ , to the bounded derived category of  $\text{Coh } X$ , denoted by  $D(X)$ .

One way to think of the notion is that it is a tool to make interesting invariants of moduli stacks, as we have seen in the foundational work [HaNa], in which the notion of *Harder-Narasimhan filtrations*, in today's term, was given birth to discuss *Tamagawa numbers* that are certain volumes of moduli spaces on curves.

We would like to recall that for D-branes in superstring theory, Douglas's work [Do] on  $\Pi$ -stabilities motivated the notion of stability conditions. In a Calabi-Yau variety  $X$ , our strings form Riemann surfaces whose boundaries restrict to subvarieties called *B-branes*, that are kinds of D-branes. With Kontsevich's framework [Ko], in  $D(X)$ , the notion of  $\Pi$ -stabilities discusses configuration of B-branes and its deformation, which is locally parameterized by *central charges* of B-branes. In this term, we are taking invariants out of B-branes whose central charges align in the complex plane.

Now, we begin to be more specific for our paper, leaving formality a bit out for later sections. For stability conditions of triangulated categories, Joyce started to extend Donaldson-Thomas invariants so that wall-crossings of stability conditions give differential equations over his invariants, which we call Joyce invariants.

A commutative  $\mathbb{Q}$  algebra  $\Lambda$  containing  $l$  and a *motivic invariant*  $I$  from the category of Artin stacks of finite type to  $\Lambda$  satisfy the following: for  $I(\mathbb{C}) = l$ , we

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have  $I(\mathrm{GL}(n, \mathbb{C})) = l^{n^2}(1 - l^{-1}) \cdots (1 - l^{-n})$  invertible in  $\Lambda$ , for quasiprojective varieties  $X$  and  $Y$ , we have  $I(X \times Y) = I(X)I(Y)$ , for a closed quasiprojective variety  $Y$  in  $X$ , we have  $I(X) = I(Y) + I(X/Y)$ , and for a quotient stack  $[X/G]$  with a special algebraic group  $G$ , which is a group embedded in some  $\mathrm{GL}(n, \mathbb{C})$  with  $\mathrm{GL}(n, \mathbb{C}) \rightarrow \mathrm{GL}(n, \mathbb{C})/G$  having locally trivial fibers, we have  $I([X/G]) = I(X)/I(G)$ .

For example, some motivic invariant extends the ring structures of Poincaré or Hodge polynomials on the category of smooth projective varieties to our category. Generally, each motivic invariant factors through the ring of isomorphism classes of above quotient stacks  $[X/G]$  [Jo07b].

From here, we will assume that  $X$  denotes an algebraic K3 surface  $X$ , and, in the stability manifold of stability conditions on  $D(X)$ ,  $\mathrm{Stab}^*(X)$  denotes the connected component constructed by Bridgeland [Br08]. For a stability condition  $\sigma$  of Gieseker on  $\mathrm{Coh} X$  or of Bridgeland on  $D(X)$  and *Mukai vectors*  $\alpha$  in the *Mukai lattice* of  $X$  [Mu], which is a nondegenerate even integer lattice, let  $M^\alpha(\sigma)$  be moduli stacks of semistable objects with respect to  $\sigma$ ; now, Joyce invariants  $J^\alpha(\sigma)$  are defined with these moduli stacks and motivic invariants.

In [Jo08], on  $\mathrm{Coh} X$ , Joyce proved that his invariants exist independently of the choice of Gieseker stability conditions; then, on  $D(X)$ , he discussed his invariants, supposing that his invariants exist independently of the choice of stability conditions in  $\mathrm{Stab}^*(X)$ , which was proved by Toda [To].

Now, with the notion of *numerically faithfulness* (*faithful* for short), which was introduced by the second author [Ok07b], for each moduli stack, the independence of the choice of stability conditions in  $\mathrm{Stab}^*(X)$  for Joyce invariants of  $D(X)$  manifests itself as follows.

**Theorem 1.1.** *For each K3 surface  $X$ , Mukai vector  $\alpha$  of  $X$ , faithful stability conditions  $\sigma, \sigma' \in \mathrm{Stab}^*(X)$ , and motivic invariant  $I$ , we have  $I(M^\alpha(\sigma)) = I(M^\alpha(\sigma'))$ .*

Here, by [Ok07b], faithful stability conditions exist as a dense subset in  $\mathrm{Stab}^*(X)$ . So, in  $\mathrm{Stab}^*(X)$ , for a set of semistable objects with a bounded mass, by wall structures examined in [Br08], for each Mukai vector  $\alpha$  and polarization of  $X$ , we have some faithful stability condition  $\sigma \in \mathrm{Stab}^*(X)$  such that  $M^\alpha(\sigma)$  consists of Gieseker semistable coherent sheaves.

One can check that Theorem 1.1 holds on any other known stability manifolds for Calabi-Yau surfaces such as abelian surfaces and minimal resolutions of surface singularities (for references of these stability manifolds, one can consult with the Bridgeland's survey [Br06]). Also, for some moduli stacks of stable objects and moduli stacks of  $\mu$ -semistable coherent sheaves on  $X$ , one can compare Theorem 1.1 with a sequence of flops in [ArBeLi] and dimension counting in [Yo].

To make explicit computation of motivic invariants, we first want to know our moduli stacks as moduli spaces in some details, and then compute isomorphism groups of objects of moduli stacks. For example, for primitive Mukai vectors of positive ranks, by [Yo], moduli spaces of Gieseker stable coherent sheaves

are deformation equivalent to Hilbert schemes, and they have the trivial  $\mathbb{C}^*$  isomorphism group for each point in the moduli stacks.

Going beyond above primitive cases is a challenge; reasons include that in general moduli spaces can be singular and computing isomorphism groups of objects is demanding. To explain what happens in a situation, for objects  $E, F$  and their Mukai vectors  $[E], [F]$ , let  $[E].[F] = \sum_i (-1)^i \dim \operatorname{Ext}^i(E, F)$  be the Mukai paring of  $[E]$  and  $[F]$ . For objects in the moduli stack of a Mukai vector with non-positive self Mukai paring, in the moduli stack of a multiple of the Mukai vector, their direct products have a nontrivial fiber with some isomorphism group for each point in the fiber.

On the other hand, for a Mukai vector with positive self Mukai paring, by [Ok07b], Theorem 1.1 boils down to Corollary 1.2. Let us recall Mukai vectors  $\alpha$  are called *spherical*, if their self-intersections are two; in other words,  $\alpha$  correspond to *spherical objects* which not only give rise to autoequivalences of  $D(X)$  [ST], but also include structure sheaves supported over rational curves on  $X$ , the structure sheaf of  $X$ , and their twists by line bundles. Notice that each Mukai vector  $v$  with  $v.v > 0$  is a multiple of a spherical Mukai vector  $\alpha$ .

**Corollary 1.2.** For each spherical class  $\alpha$ , faithful  $\sigma \in \operatorname{Stab}^*(X)$ , positive integer  $n$ , and motivic invariant  $I$ , we have  $I(M^{n\alpha}(\sigma)) = I([1/\operatorname{GL}(n, \mathbb{C})]) = \frac{1}{l^{n^2}(1-l^{-1})\dots(1-l^{-n})}$ .

In other words, for faithful stability conditions, we always have a stable spherical object for each spherical class. As pointed out to the authors by Bridgeland, the existence of a stable spherical object of each spherical class in Corollary 1.2 in particular gives another way to prove that  $\operatorname{Stab}^*(X)$  is locally a bundle over the *period domain* of  $X$ , which consists of complexified Kähler classes of  $X$  without ones that are orthogonal to spherical classes.

Once we know our moduli stacks in these details, then we are able to compute various invariants. Indeed, after the second author discussed some part of the content of this paper such as Corollary 1.3 (in the original form of Joyce invariants for some  $\alpha$ ) at [Ok07a] and whilst the authors were preparing this paper, they got notified that for standard stabilities of coherent sheaves of rational elliptic surfaces, Yoshioka–Nakajima computed their invariants [NaYo]. Also, for stability conditions of Calabi–Yau categories of dimension three (a.k.a. 3-Calabi–Yau categories), Kontsevich–Soibelman discussed their invariants [KoSo].

Here we will stick to Joyce invariants for K3 surfaces, but let us make some comments for our readers. Unlike invariants defined by Nakajima–Yoshioka, Joyce invariants involve not only arbitrary motivic invariants, but also correction terms of powers of  $q$  based on Lie algebras associated to each stability condition.

The invariants discussed by Kontsevich–Soibelman are (presumably) compatible with Joyce invariants, and they put primary emphasis on nontrivial wall-crossing formulas of their invariants for Calabi–Yau categories of dimension three.

Now, let us go back to our case; for the Joyce invariants in Corollary 1.2, we compute as below. For the convenience of our formulas, we will use  $q = l^{-1}$ , switching between *Tate motive* and *Lefschetz motive*.

**Corollary 1.3.**  $J^{n\alpha}(\sigma) = \frac{q^{n^2}}{n(1-q^n)}$ .

Here we would like to mention that we are slightly modifying the original formulation of Joyce invariants for K3 surfaces, as suggested to the authors by Zagier. Namely, in order to obtain more natural expressions, we omit the factor  $(q^{-1} - 1)$  (this is  $(l - 1)$  in [Jo08]). Recall that the factor  $(q^{-1} - 1)$  was involved so that we are able to get numbers on moduli stacks of stable objects by replacing  $q$  by one. Instead, we take residues at  $q = 1$  to extend the notion of Euler characteristics to moduli stacks, which are not necessarily only of stable objects.

We may regard Joyce invariants as volumes for each Mukai vector by the following reason. By Theorem 1.1, for each Mukai vector and generic choices of stability conditions, motivic invariants ignore the difference of moduli stacks, but unlike Joyce invariants, on deformations of stability conditions on stability manifolds, we do not know whether motivic invariants deform on moduli stacks.

So now, we would like to take the following generating functions of Joyce invariants:

$$J_k = \sum_{n>0} \frac{J^{n\alpha}(\sigma)}{n^k} = \sum_{n>0} \frac{q^{n^2}}{n^{k+1}(1-q^n)}.$$

Let us point out that taking residues termwise at  $q = 1$  gives  $-\zeta(k + 2)$ .

The generating function  $J_k$  actually appears in the following sum suggested by Joyce [Jo07b]. Namely, on a stability manifold, we can consider the form  $\sum_{\alpha \neq 0} \frac{J^{n\alpha}(\sigma)}{Z(n\alpha)^k}$ , which is invariant under autoequivalences. Also let us note that we can take the smaller form  $\sum_{\alpha, \alpha=2} \frac{J^{n\alpha}(\sigma)}{Z(n\alpha)^k}$ , which is again invariant under autoequivalences. Here we would like to study its building piece  $J_k$ . It is clear that cases of  $k$  being odd give degenerated forms; so we will concentrate on cases when  $k$  is even.

Let us also mention that by the work of Bridgeland–Toledano-Laredo [BrTo] and Kontsevich–Soibelman [KoSo], it has become clear that invariants of the moduli stacks whose images of central charges align make a building block to study Lie algebras associated to stability conditions.

As we have seen, generating functions coming out of physics have been discussed with modular forms. Now,  $J_k$  are already some quantum polylogarithms, as they are  $q$ -deformations of polylogarithms. However, the presence of  $q^{n^2}$  in the numerator does not make in particular  $J_0$  the well-known quantum dilogarithm (for example, see [Za07b]), but instead  $J_k$  look similar to some of *mock theta functions*, which were introduced by Ramanujan [Ra00] [Ra88] and carry transformation laws similar to ones of theta functions. We will pursue this view point.

Let us take a quick review at mock theta functions. The explicit definition on these functions was not given by Ramanujan, and this issue had remained for a long time. However, quite recently, Zwegers in his thesis [Zw] provided a way to add correction terms to the Ramanujan’s mock theta functions to make them into *harmonic weak Maass forms of weight  $\frac{1}{2}$*  [Za07a], which is explained as follows.

For  $\tau$  in  $q = e^{2\pi i\tau}$ , let  $\overline{D}$  be the differential operator  $\frac{1}{2\pi i} \frac{d}{d\tau}$ ,  $M^k$  be the space of meromorphic modular forms of weight  $k$  for  $k \in \frac{1}{2}\mathbb{Z}$  with poles only at the cusps, and  $\tau = x + iy$ . Then, harmonic weak Maass forms of weight  $k$  are real analytic modular forms whose derivatives with respect to  $\overline{D}$  fall into the space  $\frac{M^{2-k}}{y^k}$ ; here, these derivatives for mock theta functions are called *shadows* [Za07a].

Since this understanding of mock theta functions surfaced, we have seen achievements such as [BrOn06], [BrOn07], and [BrOn]. Especially, Fourier coefficients of harmonic weak Maass forms of weight  $\frac{1}{2}$  played a central role, in particular, for solving *the Andrews-Dragonette Conjecture*, that is to prove an exact formula of Fourier coefficients of a mock theta function.

Also, for an even integer  $k > 2$ , the first author in his thesis [Me] studied the so-called *higher Green's functions of weight  $k$* , which are directly related to the harmonic weak Maass forms of weight  $2 - k$  by the *Maass operators*  $(y^2 \overline{D})^{\frac{k-2}{2}}$  and  $(D + \frac{2}{4\pi iy}) \cdots (D + \frac{k-2}{4\pi iy})$  with the differential operator  $D = \frac{1}{2\pi i} \frac{d}{d\tau}$ .

Now, going back to our  $J_k$ , with certain duality, we want to compensate our choices of positive integers  $k$ . This can be done in terms of differential equations, modular forms, and certain correction terms to  $J_k$ . Here, differential operator  $D$  may correspond to infinitesimal derivatives of our volumes. Let  $E_k$  and  $B_k$  be the Eisenstein series and the Bernoulli numbers. Then, we have

$$D^{k-1} J_{k-2} = \frac{B_k}{2k} (1 - E_k) - J_{-k} + \sum_{n>0} \frac{q^{n^2}}{n^{1-k}}.$$

We would like to have a duality formula which contains only modular forms as follows. Let us recall that in the space of modular forms of a given degree, Eisenstein series make distinguished basis of the subspace that is orthogonal to cusp forms. Now, we take the following.

**Definition 1.4.**

$$\mathcal{J}_k = \frac{B_{-k}}{2k} - \frac{1}{2} \sum_{n>0} \frac{q^{n^2}}{n^{k+1}} + J_k = \frac{B_{-k}}{2k} + \sum_{n \neq 0} \frac{q^{n^2}}{n^{k+1}(1 - q^n)}.$$

Then, this time, for positive even integers  $k$ , we have

$$D^{k-1} \mathcal{J}_{k-2} + \mathcal{J}_{-k} = -\frac{B_k}{2k} E_k.$$

Now, we will take  $\mathcal{J}_k$  as granted, and study  $\mathcal{J}_{-2}$  in some detail.

Here,

$$\mathcal{J}_{-2}(\tau) = -\frac{1}{24} - \frac{1}{2} \sum_{n>0} nq^{n^2} + J_{-2},$$

and  $\sum_{n>0} nq^{n^2}$  is a *half-theta function*. Let  $\theta_1(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$  and  $\theta_3(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i (n + \frac{1}{2})^2 \tau}$  be half-period Jacobi theta functions (at  $z = 0$ ). Then we have the following.

**Theorem 1.5.** For  $\mathrm{SL}(2, \mathbb{Z})$ , with bounded growth at the cusp, there is a unique real analytic modular form of weight two  $\tilde{g}(\tau)$  such that the derivative of  $\tilde{g}(\tau)$  with respect to  $\overline{D}$  is  $-\frac{\theta_1(2\tau)\overline{\theta_1(2\tau)} + \theta_3(2\tau)\overline{\theta_3(2\tau)}}{64\pi^2 y^{\frac{3}{2}}}$ . Now the holomorphic part of  $\tilde{g}(\tau)$  coincides with  $\mathcal{J}_{-2}(\tau)$ .

Let us explain the words “holomorphic part” in Theorem 1.5; for holomorphic functions  $a(\tau)$  and  $b(\tau)$ , there is a canonical way to produce a function whose derivative with respect to  $\overline{D}$  is a function of the form  $\frac{a(\tau)\overline{b(\tau)}}{y^k}$ . It is given by the following integral (whenever the integral converges):

$$R\left(\frac{a(\tau)\overline{b(\tau)}}{y^k}; \tau\right) := 2\pi i a(\tau) \int_{i\infty}^{\tau} \frac{b(z)dz}{(-\frac{i}{2}(z - \bar{\tau}))^k}.$$

Now, the difference  $\tilde{g}(\tau) - R(\overline{D}(\tilde{g}(\tau)))$  vanishes by  $\overline{D}$ , and we call it *the holomorphic part*.

The story of Ramanujan’s functions is parallel to Theorem 1.5, since they can be obtained as the holomorphic parts of certain harmonic weak Maass forms of weight  $\frac{1}{2}$ . Indeed, we prove  $\mathcal{J}_{-2}(\tau)$  is in the space of mock theta functions of weight  $\frac{3}{2}$  tensored by the space  $M^{\frac{1}{2}}$ .

Let us explain a bit more. Here, the shadow is not in the space  $\frac{M^{\frac{1}{2}}}{y^{\frac{3}{2}}}$ , but in the twisted space  $\frac{M^{\frac{1}{2}} \otimes \overline{M^{\frac{1}{2}}}}{y^{\frac{3}{2}}}$ . Also, the holomorphic part of  $\tilde{g}(\tau)$  is not a mock theta function, but a sum of products of ordinary theta functions of weight  $\frac{1}{2}$  and mock theta functions of weight  $\frac{3}{2}$ , which will be derived in this paper from the *Lerch function* in [Zw]. We will then be able to identify the Fourier coefficients of the sum to end the proof of Theorem 1.5.

Now, authors are aware that we are leaving many questions open. For example, we would want some understanding of moduli stacks of cases other than ones considered here and  $\mathcal{J}_k$  for  $k \neq -2$ , but it is our impression that they rather pose fundamental questions on isomorphism groups of points in moduli stacks, algebras on moduli stacks, and mock theta functions.

Yet, here, we investigated our cases in some detail and thank the Dyson’s dream [Dy, Section 6], which at some point encouraged us to look for mock symmetries in this context.

## 2 Definitions

Let us recall fundamental notions from [Br07]. In this paper, our triangulated category  $\mathcal{T}$  is assumed to be  $\mathrm{D}(X)$  for some K3 surface  $X$ . Let  $K(\mathcal{T})$  be the *Grothendieck group* of  $\mathcal{T}$ ; i.e.,  $K(\mathcal{T})$  is the abelian group generated by classes of objects of  $\mathcal{T}$  such that for objects  $E, F, G$  in  $\mathcal{T}$ , we have  $[F] = [E] + [G]$  in  $K(\mathcal{T})$  whenever we have an exact triangle  $E \rightarrow F \rightarrow G$  in  $\mathcal{T}$ .

## 2.1 Stability conditions

A *stability condition*  $\sigma = (Z, \mathcal{P})$  on  $\mathcal{T}$  consists of a group homomorphism  $Z$  from  $K(\mathcal{T})$  to the complex number  $\mathbb{C}$  and a family  $\mathcal{P}(\phi)$  of full abelian subcategories of  $\mathcal{T}$  indexed by real numbers  $\phi$ . Each  $Z$  and  $\mathcal{P}$  are called a *central charge* and a *slicing*. They need to satisfy the following compatibilities.

- If for some  $\phi \in \mathbb{R}$ ,  $E$  is a nonzero object in  $\mathcal{P}(\phi)$ , then for some positive real number  $m(E)$ , called *mass* of  $E$ , we have  $Z(E) = m(E) \exp(i\pi\phi)$ .
- For each real number  $\phi$ , we have  $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ .
- For real numbers  $\phi_1 > \phi_2$  and objects  $A_i \in \mathcal{P}(\phi_i)$ , we have  $\text{Hom}_{\mathcal{T}}(A_1, A_2) = 0$ .
- For any nonzero object  $E \in \mathcal{T}$ , there exist real numbers  $\phi_1 > \dots > \phi_n$  and objects  $A_i \in \mathcal{P}(\phi_i)$  such that there exists a sequence of exact triangles  $E_{i-1} \rightarrow E_i \rightarrow A_i$  with  $E_0 = 0$  and  $E$ .

The sequence above is called the *Harder-Narasimhan filtration* (*HN-filtration* for short) of  $E$ . The HN-filtration of any object is unique up to isomorphisms. For each  $\phi \in \mathbb{R}$ , nonzero objects in  $\mathcal{P}(\phi)$  are called *semistable* with *phase*  $\phi$ . If moreover a semistable object in  $\mathcal{P}(\phi)$  has only the trivial Jordan-Hölder filtration in  $\mathcal{P}(\phi)$ , then it is called *stable*.

We will assume that our central charge  $Z$  factors through the map  $[E] \in K(\mathcal{T}) \mapsto \text{ch}(E)\sqrt{X}$ , which is the *Mukai vector* of  $E$  in the *Mukai lattice* of  $X$ . For Mukai vectors  $v, w$ , let  $v.w$  be the *Mukai paring*.

A stability condition  $\sigma = (Z, \mathcal{P})$  is called *numerically faithful* [Ok07b, Definition 3.1] (*faithful* for short), if for each real number  $r$ , we have a primitive Mukai vector  $v$  such that for each semistable object  $E$  of the phase  $r$ ,  $[E]$  is a sum of  $v$ . Here, by [Br08, Proposition 8.3], the connected component  $\text{Stab}^*(X)$  satisfies the assumption of [Ok07b, Lemma 3.1]. So faithful stability conditions are dense in  $\text{Stab}^*(X)$ .

For a real number  $r$ , let  $\mathcal{P}(r - 1, r]$  be the extension-closed full subcategory consisting of semistable objects whose phases are in the interval  $(r - 1, r]$ , and  $C(r)$  be the Mukai vectors of the objects in  $\mathcal{P}(r - 1, r]$ . Then, for each stability condition  $\sigma = (Z, \mathcal{P}) \in \text{Stab}^*(X)$ , Mukai vector  $\alpha$ , and the real number  $r$  such that  $Z(\alpha) \in \mathbb{R}_{>0} e^{i\pi r}$ , we define the Joyce invariant  $J^\alpha(\sigma)$  to be  $\sum_{n=1}^{\infty} \sum_{\alpha_1 + \dots + \alpha_n = \alpha, \alpha_i \in C(r)} q^{\sum_{j>i} \alpha_j \cdot \alpha_i} \frac{(-1)^{n-1}}{n} \prod_{i=1}^n I(M^{\alpha_i}(\sigma))$  [Jo08, Definition 6.22], [To, Definition 5.9] (let us recall that as explained in the introduction, we let  $q = l^{-1}$  and omit  $(l - 1)$  from their original definitions).

## 2.2 Modular forms

Let us recall the definition and properties of the Dedekind eta function (we denote  $q = e^{2\pi i\tau}$ ),  $\tau$  belongs to the upper half plane.

$$\eta(\tau) = e^{\frac{\pi i\tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau}) = q^{1/24} (1 - q - q^2 + q^5 + q^7 \dots).$$

The eta functions transforms like a modular form of weight  $\frac{1}{2}$ :

$$\eta(\tau + 1) = e^{\frac{\pi i}{12}} \eta(\tau), \quad \eta\left(\frac{-1}{\tau}\right) = e^{\frac{\pi i}{4}} \sqrt{\tau} \eta(\tau).$$

We will need the following identity:

$$\eta\left(\frac{\tau}{2}\right) \eta\left(\frac{1+\tau}{2}\right) \eta(2\tau) = e^{\frac{\pi i}{24}} \eta(\tau)^3. \quad (1)$$

Next we recall the half-period Jacobi theta functions (at  $z = 0$ ), note that we slightly changed the indexing:

$$\begin{aligned} \theta_1(\tau) &= \theta_{00}(0; \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau} &= 1 + 2q^{\frac{1}{2}} + 2q^2 + 2q^{\frac{9}{2}} + 2q^8 + \dots, \\ \theta_2(\tau) &= \theta_{01}(0; \tau) = \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i n^2 \tau} &= 1 - 2q^{\frac{1}{2}} + 2q^2 - 2q^{\frac{9}{2}} + 2q^8 + \dots, \\ \theta_3(\tau) &= \theta_{10}(0; \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i (n + \frac{1}{2})^2 \tau} &= 2q^{\frac{1}{8}} + 2q^{\frac{9}{8}} + 2q^{\frac{25}{8}} + 2q^{\frac{49}{8}} + \dots. \end{aligned}$$

The theta functions can be expressed in terms of the eta function in the following way:

$$\theta_1(\tau) = e^{-\frac{\pi i}{12}} \frac{\eta\left(\frac{1+\tau}{2}\right)^2}{\eta(\tau)}, \quad \theta_2(\tau) = \frac{\eta\left(\frac{\tau}{2}\right)^2}{\eta(\tau)}, \quad \theta_3(\tau) = 2 \frac{\eta(2\tau)^2}{\eta(\tau)}. \quad (2)$$

We know their transformation properties:

$$\begin{aligned} \theta_1(\tau + 1) &= \theta_2(\tau), & \theta_1\left(-\frac{1}{\tau}\right) &= e^{-\frac{\pi i}{4}} \sqrt{\tau} \theta_1(\tau), \\ \theta_2(\tau + 1) &= \theta_1(\tau), & \theta_2\left(-\frac{1}{\tau}\right) &= e^{-\frac{\pi i}{4}} \sqrt{\tau} \theta_3(\tau), \\ \theta_3(\tau + 1) &= e^{\frac{\pi i}{4}} \theta_3(\tau), & \theta_3\left(-\frac{1}{\tau}\right) &= e^{-\frac{\pi i}{4}} \sqrt{\tau} \theta_2(\tau). \end{aligned}$$

In particular, they are modular forms for the group  $\Gamma(2)$ .

We also need the classical Eisenstein series of weight 2 for  $\text{SL}(2, \mathbb{Z})$ :

$$E_2(\tau) = 24 \frac{\eta'(\tau)}{\eta(\tau)} = 1 - 24 \sum_{k, n=1}^{\infty} k q^{nk} = 1 - 24q - 72q^2 - 96q^3 - \dots.$$

The function  $E_2$  is not a modular form, but is quasi-modular form. The fourth powers of the theta functions are Eisenstein series for  $\Gamma(2)$  and we have



the following relations:

$$E_2\left(\frac{1+\tau}{2}\right) - 2E_2(\tau) = \theta_1^4(\tau) - 2\theta_2(\tau)^4, \quad (3)$$

$$E_2\left(\frac{\tau}{2}\right) - 2E_2(\tau) = -2\theta_1^4(\tau) + \theta_2(\tau)^4, \quad (4)$$

$$4E_2(2\tau) - 2E_2(\tau) = \theta_1^4(\tau) + \theta_2(\tau)^4, \quad (5)$$

$$\theta_1^4(\tau) = \theta_2^4(\tau) + \theta_3^4(\tau). \quad (6)$$

### 2.3 The Lerch function

Having introduced some classical modular forms, we turn to the thesis of Zwegers [Zw]. In this thesis we find the following definition of the Lerch function:

$$\mu(u, v; \tau) = \frac{e^{\pi i u}}{\theta(v; \tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{\pi i (n^2 + n)\tau + 2\pi i n v}}{1 - e^{2\pi i n \tau + 2\pi i u}} \quad (u, v \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})).$$

The definition of the theta function he uses is the following one:

$$\theta(z; \tau) = \sum_{\nu \in \frac{1}{2} + \mathbb{Z}} e^{\pi i \nu^2 \tau + 2\pi i \nu (z + \frac{1}{2})}.$$

Note the following symmetry:

$$\theta(z+1; \tau) = \theta(-z; \tau) = -\theta(z; \tau), \quad \theta(z+\tau; \tau) = -e^{-\pi i \tau - 2\pi i z} \theta(z; \tau).$$

The theta functions  $\theta_1$ ,  $\theta_2$  and  $\theta_3$  are related to  $\theta$  in the following way:

$$\begin{aligned} \theta\left(\frac{\tau}{2}; \tau\right) &= -ie^{-\frac{\pi i \tau}{4}} \theta_2(\tau), \\ \theta\left(\frac{1+\tau}{2}; \tau\right) &= -e^{-\frac{\pi i \tau}{4}} \theta_1(\tau), \\ \theta\left(\frac{1}{2}; \tau\right) &= -\theta_3(\tau). \end{aligned}$$

Moreover we have

$$\theta(0; \tau) = 0, \quad \left. \frac{d}{2\pi i ds} \right|_{s=0} \theta(s; \tau) = i\eta^3(\tau).$$

Zwegers found a way to add a correction term to  $\mu$  so that the new function  $\tilde{\mu}$  has good transformation properties. Namely, he defines

$$\tilde{\mu}(u, v; \tau) = \mu(u, v; \tau) + \frac{i}{2} R(u - v; \tau),$$

where

$$R(u; \tau) = \sum_{\nu \in \frac{1}{2} + \mathbb{Z}} \left\{ \text{sign}(\nu) - E\left(\left(\nu + \frac{\Im u}{y}\right) \sqrt{2y}\right) \right\} (-1)^{\nu - \frac{1}{2}} e^{-\pi i \nu^2 \tau - 2\pi i \nu u}.$$

Here  $y = \Im \tau$  and  $E$  is the function

$$E(z) = 2 \int_0^z e^{-\pi t^2} dt = 1 - \operatorname{erfc}(z\sqrt{\pi}).$$

The result of Zwegers is the following transformation properties of  $\tilde{\mu}$ :

**Theorem 2.1.** *[Zw, Theorem 1.11] The function  $\tilde{\mu}$  satisfies*

$$\tilde{\mu}(u, v; \tau) = \tilde{\mu}(v, u; \tau) = \tilde{\mu}(-u, -v; \tau),$$

and

$$\begin{aligned} \tilde{\mu}(u, v; \tau + 1) &= e^{-\frac{\pi i}{4}} \tilde{\mu}(u, v; \tau), & \tilde{\mu}\left(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\right) &= -e^{-\frac{\pi i}{4} - \frac{\pi i(u-v)^2}{\tau}} \sqrt{\tau} \tilde{\mu}(u, v; \tau) \\ \tilde{\mu}(u + 1, v; \tau) &= -\tilde{\mu}(u, v; \tau), & \tilde{\mu}(u + \tau, v; \tau) &= -e^{2\pi i(u-v) + \pi i \tau} \tilde{\mu}(u, v). \end{aligned}$$

Here is a list of properties that the functions  $R$  and  $\mu$  satisfy separately:

**Proposition 2.2.** *[Zw, Propositions 1.4 and 1.9] The functions  $\mu$  and  $R$  satisfy*

$$\mu(u, v; \tau) = \mu(v, u; \tau) = \mu(-u, -v; \tau), \quad R(-z; \tau) = R(z; \tau),$$

and we have

$$\mu(u + 1, v; \tau) = -\mu(u, v; \tau), \quad R(z + 1; \tau) = -R(z; \tau).$$

We also mention one last property which we will use:

**Proposition 2.3.** *[Zw, Proposition 1.4 and Theorem 1.11] Both the function  $\mu$  and  $\tilde{\mu}$  (if you plug it in place of  $\mu$ ) satisfy*

$$\mu(u + z, v + z; \tau) - \mu(u, v; \tau) = \frac{i\eta^3(\tau)\theta(u + v + z; \tau)\theta(z; \tau)}{\theta(u; \tau)\theta(v; \tau)\theta(u + z; \tau)\theta(v + z; \tau)}$$

for  $u, v, u + z, v + z \notin \mathbb{Z} + \tau\mathbb{Z}$ .

### 3 Proofs

Let us prove Theorem 1.1.

*Proof.* For faithful stability conditions  $\sigma$ , in terms of Mukai vectors,  $J^\alpha(\sigma)$  admit unique expressions. Since by [To, Theorem 1.5], we have  $J^\alpha(\sigma) = J^\alpha(\sigma')$  for any  $\alpha$ , especially for primitive ones, the statement follows.  $\square$

In terms of faithful stability conditions over integer lattices, for invariants of moduli stacks of aligned central charges, Theorem 1.1 is a general feature of their deformation invariance on stability manifolds.

We will prove Corollary 1.2. Now, an object  $E \in \mathcal{T}$  is called *spherical* if  $\operatorname{Ext}^i(E, E) = \mathbb{C}$  for  $i = 0, 2$  and  $\operatorname{Ext}^i(E, E) = 0$  for else; spherical classes are Mukai vectors of spherical objects.

*Proof.* For the case when  $\alpha$  is with a nonzero rank, by [Yo, Theorem 0.1(1)] and [Br08, Proposition 14.2], for some faithful  $\sigma \in \text{Stab}^*(X)$ , we have a stable spherical object whose class is  $\alpha$ . So, by [Ok07b, Proposition 4.9], the statement follows. For other cases, by [Fr, Lemma 25], the first Chern class of  $\alpha$  is either effective or anti-effective. So, by replacing  $\alpha$  with  $-\alpha$ , if necessarily, one recalls that some coherent sheaf  $E$  with  $[E] = \alpha$  is Gieseker semistable. Then, by [To, Theorem 6.6], the statement follows.  $\square$

Let us prove Corollary 1.3.

*Proof.* Since  $\alpha \cdot \alpha = 2$ , by choosing  $\sigma$  to be faithful, we have that for positive integers  $k_i$ ,  $J^{n\alpha}(\sigma)$  is equal to  $\sum_{m=1}^{\infty} \sum_{k_1+\dots+k_m=n} q^{\sum_{i>j} 2k_i k_j} \frac{(-1)^{n-1}}{n} \prod_{i=1}^n \frac{1}{I(\text{GL}(k_i, \mathbb{C}))}$ . Since  $\sum_{i>j} 2k_i k_j = (\sum k_i)^2 - \sum k_i^2 = n^2 - \sum k_i^2$ , we have that  $J^{n\alpha}(\sigma)$  is equal to  $q^{n^2} \sum_{m=1}^{\infty} \sum_{k_1+\dots+k_m=n} \frac{(-1)^{n-1}}{n} \prod_{i=1}^n \frac{q^{-k_i^2}}{I(\text{GL}(k_i, \mathbb{C}))}$ .

Let  $F(x) = \sum_{m \geq 0} \frac{q^{-m^2}}{I(\text{GL}(m, \mathbb{C}))} x^m$ . Then we have  $F(x) - F(qx) = xF(x)$ , and  $J^{n\alpha}$  is the  $n$ -th coefficient of  $q^{n^2} \sum \frac{(-1)^{n-1}}{n} (F(x) - 1) = q^{n^2} \log F(x)$ . Since  $\log F(x) + \log(1-x) = \log F(qx)$ , the  $n$ -th coefficient of  $\log F(x)$  is  $\frac{1}{n(1-q^n)}$ . So the statement follows.  $\square$

The rest of this section is devoted to the proof of Theorem 1.5. The plan is to see the existence of a function with the holomorphic part being  $\mathcal{J}_{-2}$  and good transformation properties. Now, the first clue is to notice that  $\mathcal{J}_{-2}$  looks similar to  $\mu$ , which is the holomorphic part of  $\tilde{\mu}$ , but to be precise, we will here derive several functions from  $\mu$  and subsequently modify them with theta functions.

Let us study behavior of the functions  $\mu, \tilde{\mu}, R$  at the “points of order two”. The values at these points are not interesting since we have

**Proposition 3.1.**

$$\tilde{\mu}\left(\frac{1}{2}, \frac{\tau}{2}; \tau\right) = \tilde{\mu}\left(\frac{1}{2}, \frac{1+\tau}{2}; \tau\right) = \tilde{\mu}\left(\frac{\tau}{2}, \frac{1+\tau}{2}; \tau\right) = 0.$$

*Proof.* We simply take the definition of  $\mu$  and  $R$  above and use the following trick. For example, in the case of  $\tilde{\mu}\left(\frac{1}{2}, \frac{\tau}{2}\right)$  the trick is to write

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{\pi i(n^2+2n)\tau}}{1 + e^{2\pi i n \tau}} &= \frac{1}{2} \left( \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{\pi i(n^2+2n)\tau}}{1 + e^{2\pi i n \tau}} + \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{\pi i(n^2-2n)\tau}}{1 + e^{-2\pi i n \tau}} \right) \\ &= \frac{1}{2} \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i n^2 \tau} = \frac{\theta_2(\tau)}{2}. \end{aligned}$$

Therefore for  $\tilde{\mu}\left(\frac{1}{2}, \frac{\tau}{2}\right)$  we obtain

$$\mu\left(\frac{1}{2}, \frac{\tau}{2}; \tau\right) = -\frac{e^{\frac{\pi i \tau}{4}}}{2}.$$

A trick similar to the one used above gives

$$R\left(\frac{1-\tau}{2}; \tau\right) = -ie^{\frac{\pi i \tau}{4}}.$$

Thus  $\tilde{\mu}\left(\frac{1}{2}, \frac{\tau}{2}; \tau\right) = 0$ . The other cases are similar with

$$\begin{aligned} \mu\left(\frac{1}{2}, \frac{1+\tau}{2}; \tau\right) &= -\frac{ie^{\frac{\pi i \tau}{4}}}{2}, & \mu\left(\frac{\tau}{2}, \frac{1+\tau}{2}; \tau\right) &= 0, \\ R\left(-\frac{\tau}{2}; \tau\right) &= e^{\frac{\pi i \tau}{4}}, & R\left(-\frac{1}{2}; \tau\right) &= 0. \end{aligned}$$

□

Because of the last proposition the derivatives of  $\tilde{\mu}$  at the points of order 2 should have nice transformation properties. Namely, we define

$$\begin{aligned} \mu_1(\tau) &= \frac{d}{2\pi i ds} \Big|_{s=0} \tilde{\mu}\left(\frac{1}{2}, \frac{\tau}{2} + s; \tau\right), & \mu'_1(\tau) &= \frac{d}{2\pi i ds} \Big|_{s=0} \tilde{\mu}\left(\frac{1}{2} + s, \frac{\tau}{2}; \tau\right), \\ \mu_2(\tau) &= \frac{d}{2\pi i ds} \Big|_{s=0} \tilde{\mu}\left(\frac{1}{2}, \frac{1+\tau}{2} + s; \tau\right), & \mu'_2(\tau) &= \frac{d}{2\pi i ds} \Big|_{s=0} \tilde{\mu}\left(\frac{1}{2} + s, \frac{1+\tau}{2}; \tau\right), \\ \mu_3(\tau) &= \frac{d}{2\pi i ds} \Big|_{s=0} \tilde{\mu}\left(\frac{\tau}{2}, \frac{1+\tau}{2} + s; \tau\right), & \mu'_3(\tau) &= \frac{d}{2\pi i ds} \Big|_{s=0} \tilde{\mu}\left(\frac{\tau}{2} + s, \frac{1+\tau}{2}; \tau\right). \end{aligned}$$

**Proposition 3.2.** *We have*

$$\begin{aligned} \mu_1(\tau) + \mu'_1(\tau) &= -\frac{e^{\frac{\pi i \tau}{4}} \theta_1(\tau)^3}{4}, \\ \mu_2(\tau) + \mu'_2(\tau) &= -\frac{ie^{\frac{\pi i \tau}{4}} \theta_2(\tau)^3}{4}, \\ \mu_3(\tau) + \mu'_3(\tau) &= -\frac{\theta_3(\tau)^3}{4}. \end{aligned}$$

*Proof.* Using Proposition 2.3 we obtain

$$\begin{aligned} \mu_1(\tau) + \mu'_1(\tau) &= \frac{d}{2\pi i ds} \Big|_{s=0} \left( \tilde{\mu}\left(\frac{1}{2}, \frac{\tau}{2} + s\right) - \tilde{\mu}\left(\frac{1}{2} - s, \frac{\tau}{2}\right) \right) \\ &= \frac{d}{2\pi i ds} \Big|_{s=0} \frac{i\eta(\tau)^3 \theta\left(\frac{1+\tau}{2}; \tau\right) \theta(s; \tau)}{\theta\left(\frac{1}{2} - s; \tau\right) \theta\left(\frac{1}{2}; \tau\right) \theta\left(\frac{\tau}{2} - s; \tau\right) \theta\left(\frac{\tau}{2}; \tau\right)} \\ &= \frac{-\eta(\tau)^6 \theta\left(\frac{1+\tau}{2}; \tau\right)}{\theta\left(\frac{1}{2}; \tau\right)^2 \theta\left(\frac{\tau}{2}; \tau\right)^2} = \frac{-\eta^6(\tau) e^{\frac{\pi i \tau}{4}} \theta_1(\tau)}{\theta_2(\tau)^2 \theta_3(\tau)^2}. \end{aligned}$$

We have (using (2) and (1))

$$\theta_1(\tau) \theta_2(\tau) \theta_3(\tau) = 2\eta(\tau)^3.$$

Therefore

$$\mu_1(\tau) + \mu'_1(\tau) = -\frac{e^{\frac{\pi i \tau}{4}} \theta_1(\tau)^3}{4}.$$

Similarly,

$$\begin{aligned}\mu_2(\tau) + \mu'_2(\tau) &= \frac{-\eta(\tau)^6 \theta\left(\frac{\tau}{2} + 1; \tau\right)}{\theta\left(\frac{1}{2}; \tau\right)^2 \theta\left(\frac{1+\tau}{2}; \tau\right)^2} = \frac{-i\eta^6(\tau) e^{\frac{\pi i \tau}{4}} \theta_1(\tau)}{\theta_2(\tau)^2 \theta_3(\tau)^2} = -\frac{ie^{\frac{\pi i \tau}{4}} \theta_2(\tau)^3}{4}, \\ \mu_3(\tau) + \mu'_3(\tau) &= \frac{-\eta(\tau)^6 \theta\left(\frac{1}{2} + \tau; \tau\right)}{\theta\left(\frac{\tau}{2}; \tau\right)^2 \theta\left(\frac{1+\tau}{2}; \tau\right)^2} = -\frac{\eta(\tau)^6 \theta_3(\tau)}{\theta_1(\tau)^2 \theta_2(\tau)^2} = -\frac{\theta_3(\tau)^3}{4}.\end{aligned}$$

□

Now we are ready to formulate the transformation properties of  $\mu_i$ .

**Proposition 3.3.** *We have*

$$\begin{aligned}\mu_1(1 + \tau) &= e^{-\frac{\pi i}{4}} \mu_2(\tau), & \mu_2(1 + \tau) &= -e^{-\frac{\pi i}{4}} \mu_1(\tau), \\ \mu_3(1 + \tau) &= e^{-\frac{\pi i}{4}} \left( \frac{\theta_3(\tau)^3}{4} + \mu_3(\tau) \right), \\ \mu_1\left(-\frac{1}{\tau}\right) &= e^{\frac{\pi i}{4} - \pi i \frac{1+\tau^2}{4\tau}} \tau^{\frac{3}{2}} \left( \frac{e^{\frac{\pi i \tau}{4}} \theta_1(\tau)^3}{4} + \mu_1(\tau) \right), \\ \mu_2\left(-\frac{1}{\tau}\right) &= e^{-\frac{\pi i}{4} - \frac{\pi i}{4\tau}} \tau^{\frac{3}{2}} \mu_3(\tau), & \mu_3\left(-\frac{1}{\tau}\right) &= -e^{-\frac{\pi i}{4} - \frac{\pi i \tau}{4}} \tau^{\frac{3}{2}} \mu_2(\tau).\end{aligned}$$

*Proof.* The proof of the first three equations goes by applying the operator  $\frac{d}{2\pi i ds}\big|_{s=0}$  to both sides of the following equations obtained from Theorem 2.1.

$$\begin{aligned}\tilde{\mu}\left(\frac{1}{2}, \frac{1+\tau}{2} + s; \tau + 1\right) &= e^{-\frac{\pi i}{4}} \tilde{\mu}\left(\frac{1}{2}, \frac{1+\tau}{2} + s; \tau\right), \\ \tilde{\mu}\left(\frac{1}{2}, \frac{\tau}{2} + 1 + s; \tau + 1\right) &= -e^{-\frac{\pi i}{4}} \tilde{\mu}\left(\frac{1}{2}, \frac{\tau}{2} + s; \tau\right), \\ \tilde{\mu}\left(\frac{1+\tau}{2}, \frac{\tau}{2} + 1 + s; \tau + 1\right) &= -e^{-\frac{\pi i}{4}} \tilde{\mu}\left(\frac{\tau}{2} + s, \frac{1+\tau}{2}; \tau\right).\end{aligned}$$

Similarly, for the last three equations we use

$$\begin{aligned}\tilde{\mu}\left(\frac{1}{2}, -\frac{1}{2\tau} + s; -\frac{1}{\tau}\right) &= -e^{-\frac{\pi i}{4} - \frac{\pi i(\tau+1-2s\tau)^2}{4\tau}} \sqrt{\tau} \tilde{\mu}\left(\frac{\tau}{2}, -\frac{1}{2} + s\tau; \tau\right), \\ \tilde{\mu}\left(\frac{1}{2}, \frac{\tau-1}{2\tau} + s; -\frac{1}{\tau}\right) &= -e^{-\frac{\pi i}{4} - \frac{\pi i(1-2s\tau)^2}{4\tau}} \sqrt{\tau} \tilde{\mu}\left(\frac{\tau}{2}, \frac{\tau-1}{2} + s\tau; \tau\right), \\ \tilde{\mu}\left(-\frac{1}{2\tau}, \frac{\tau-1}{2\tau} + s; -\frac{1}{\tau}\right) &= -e^{-\frac{\pi i}{4} - \frac{\pi i(\tau+2s\tau)^2}{4\tau}} \sqrt{\tau} \tilde{\mu}\left(-\frac{1}{2}, \frac{\tau-1}{2} + s\tau; \tau\right).\end{aligned}$$

□

Looking at the proposition above it is clear that we should consider the following three functions

$$\begin{aligned}\tilde{h}_1(\tau) &= e^{-\frac{\pi i \tau}{4}} \mu_1(\tau) + \frac{\theta_1(\tau)^3}{8}, \\ \tilde{h}_2(\tau) &= -ie^{-\frac{\pi i \tau}{4}} \mu_2(\tau) + \frac{\theta_2(\tau)^3}{8}, \\ \tilde{h}_3(\tau) &= -\mu_3(\tau) - \frac{\theta_3(\tau)^3}{8}.\end{aligned}$$

Then we have

$$\begin{aligned}\tilde{h}_1(\tau + 1) &= \tilde{h}_2(\tau), & \tilde{h}_1\left(-\frac{1}{\tau}\right) &= e^{\frac{\pi i}{4}} \tau^{\frac{3}{2}} \tilde{h}_1(\tau), \\ \tilde{h}_2(\tau + 1) &= \tilde{h}_1(\tau), & \tilde{h}_2\left(-\frac{1}{\tau}\right) &= e^{\frac{\pi i}{4}} \tau^{\frac{3}{2}} \tilde{h}_3(\tau), \\ \tilde{h}_3(\tau + 1) &= e^{-\frac{\pi i}{4}} \tilde{h}_3(\tau), & \tilde{h}_3\left(-\frac{1}{\tau}\right) &= e^{\frac{\pi i}{4}} \tau^{\frac{3}{2}} \tilde{h}_2(\tau).\end{aligned}$$

Next we need to find the Fourier expansions of  $\tilde{h}_i$ . We would like to have them similar to the decomposition  $\tilde{\mu} = \mu + \frac{i}{2}R$ . Therefore we compute

**Proposition 3.4.**

$$\begin{aligned}\frac{d}{2\pi i ds} \Big|_{s=0} R\left(\frac{1-\tau}{2} - s; \tau\right) &= -ie^{\frac{\pi i \tau}{4}} \left( \sum_{n \in \mathbb{Z}} |n| \beta(2yn^2) e^{-\pi i n^2 \tau} + \frac{1}{2} - \frac{\overline{\theta_1(\tau)}}{\pi \sqrt{2y}} \right), \\ \frac{d}{2\pi i ds} \Big|_{s=0} R\left(-\frac{\tau}{2} - s; \tau\right) &= e^{\frac{\pi i \tau}{4}} \left( \sum_{n \in \mathbb{Z}} (-1)^n |n| \beta(2yn^2) e^{-\pi i n^2 \tau} + \frac{1}{2} - \frac{\overline{\theta_2(\tau)}}{\pi \sqrt{2y}} \right), \\ \frac{d}{2\pi i ds} \Big|_{s=0} R\left(-\frac{1}{2} - s; \tau\right) &= i \left( \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} |\nu| \beta(2y\nu^2) e^{-\pi i \nu^2 \tau} - \frac{\overline{\theta_3(\tau)}}{\pi \sqrt{2y}} \right).\end{aligned}$$

*Proof.* Differentiating term by term gives

$$\begin{aligned}\frac{d}{2\pi i ds} \Big|_{s=0} R(u - s; \tau) &= \sum_{\nu \in \frac{1}{2} + \mathbb{Z}} \nu \left\{ \text{sign}(\nu) - E\left(\left(\nu + \frac{\Im u}{y}\right) \sqrt{2y}\right) \right\} (-1)^{\nu - \frac{1}{2}} e^{-\pi i \nu^2 \tau - 2\pi i \nu u} \\ &\quad - \sum_{\nu \in \frac{1}{2} + \mathbb{Z}} \frac{1}{\pi \sqrt{2y}} (-1)^{\nu - \frac{1}{2}} e^{-2\pi y \left(\nu + \frac{\Im u}{y}\right)^2 - \pi i \nu^2 \tau - 2\pi i \nu u}.\end{aligned}$$

Therefore for the first case we obtain

$$-ie^{\frac{\pi i \tau}{4}} \sum_{n \in \mathbb{Z}} \left(n + \frac{1}{2}\right) \left(\text{sign}\left(n + \frac{1}{2}\right) - E(n\sqrt{2y})\right) e^{-\pi i n^2 \tau} + \frac{ie^{\frac{\pi i \tau}{4}}}{\pi\sqrt{2y}} \sum_{n \in \mathbb{Z}} e^{-\pi i n^2 \bar{\tau}}.$$

The first summand can be transformed into

$$-ie^{\frac{\pi i \tau}{4}} \sum_{n \in \mathbb{Z}} \left(n + \frac{1}{2}\right) (\text{sign}(n) - E(n\sqrt{2y})) e^{-\pi i n^2 \tau} - \frac{ie^{\frac{\pi i \tau}{4}}}{2}.$$

Using the function  $\beta$ ,

$$\beta(x) = \sum_x^\infty t^{-\frac{1}{2}} e^{-\pi t} dt = 1 - E(\sqrt{x}) = \text{erfc}(\sqrt{\pi x}),$$

we obtain

$$\begin{aligned} & \frac{d}{2\pi i ds} \Big|_{s=0} R\left(\frac{1-\tau}{2} - s; \tau\right) \\ &= -ie^{\frac{\pi i \tau}{4}} \sum_{n \in \mathbb{Z}} \left(n + \frac{1}{2}\right) \text{sign}(n) \beta(2yn^2) e^{-\pi i n^2 \tau} - \frac{ie^{\frac{\pi i \tau}{4}}}{2} + \frac{ie^{\frac{\pi i \tau}{4}}}{\pi\sqrt{2y}} \overline{\theta_1(\tau)}, \end{aligned}$$

and the final result easily follows from this formula.

Analogously, in the second case we obtain

$$\begin{aligned} & \frac{d}{2\pi i ds} \Big|_{s=0} R\left(-\frac{\tau}{2} - s; \tau\right) \\ &= e^{\frac{\pi i \tau}{4}} \sum_{n \in \mathbb{Z}} (-1)^n \left(n + \frac{1}{2}\right) \text{sign}(n) \beta(2yn^2) e^{-\pi i n^2 \tau} + \frac{e^{\frac{\pi i \tau}{4}}}{2} - \frac{e^{\frac{\pi i \tau}{4}}}{\pi\sqrt{2y}} \overline{\theta_2(\tau)}, \end{aligned}$$

In the third case the result follows right from the following formula:

$$\frac{d}{2\pi i ds} \Big|_{s=0} R\left(-\frac{1}{2} - s; \tau\right) = i \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} |\nu| \beta(2y\nu^2) e^{-\pi i \nu^2 \tau} - i \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} \frac{e^{-\pi i \nu^2 \bar{\tau}}}{\pi\sqrt{2y}}.$$

□

It remains to differentiate the function  $\mu$ .

**Proposition 3.5.** *The corresponding derivatives of  $\mu$  are given by*

$$\begin{aligned}
\frac{d}{2\pi i ds} \Big|_{s=0} \mu \left( \frac{1}{2}, \frac{\tau}{2} + s; \tau \right) &= -\frac{e^{\frac{\pi i \tau}{4}}}{24\theta_1(\tau)} \left( -2 + 6\theta_1(\tau) + 3\theta_1^4(\tau) - E_2(\tau) + 48 \sum_{n=1}^{\infty} \frac{e^{\pi i(n^2+2n)\tau}}{(1 - e^{2\pi i n \tau})^2} \right), \\
\frac{d}{2\pi i ds} \Big|_{s=0} \mu \left( \frac{1}{2}, \frac{1+\tau}{2} + s; \tau \right) &= -i \frac{e^{\frac{\pi i \tau}{4}}}{24\theta_2(\tau)} \left( -2 + 6\theta_2(\tau) + 3\theta_2^4(\tau) - E_2(\tau) + 48 \sum_{n=1}^{\infty} \frac{(-1)^n e^{\pi i(n^2+2n)\tau}}{(1 - e^{2\pi i n \tau})^2} \right), \\
\frac{d}{2\pi i ds} \Big|_{s=0} \mu \left( \frac{\tau}{2}, \frac{1+\tau}{2} + s; \tau \right) &= \frac{1}{24\theta_3(\tau)} \left( 1 - 3\theta_3(\tau)^4 - E_2(\tau) + 24 \sum_{n=1}^{\infty} e^{\pi i(n^2+n)\tau} \frac{1 + e^{2\pi i n \tau}}{(1 - e^{2\pi i n \tau})^2} \right).
\end{aligned}$$

*Proof.* We use the following decomposition of  $\mu$ :

$$\begin{aligned}
\mu(s, z; \tau) &= \frac{e^{\pi i s}}{\theta(z; \tau)(1 - e^{2\pi i s})} \\
&+ \frac{1}{\theta(z; \tau)} \sum_{n=1}^{\infty} (-1)^n e^{\pi i(n^2+n)\tau} \left( \frac{e^{2\pi i n z + \pi i s}}{1 - e^{2\pi i n \tau + 2\pi i s}} - \frac{e^{-2\pi i n z - \pi i s}}{1 - e^{2\pi i n \tau - 2\pi i s}} \right).
\end{aligned}$$

We compute the Taylor expansion with respect to  $2\pi i s$  around  $s = 0$  of the expression above for the following values of  $z$ :  $\frac{1+\tau}{2}$ ,  $\frac{\tau}{2}$ ,  $\frac{1}{2}$ . We need only the coefficient at  $2\pi i s$ . This coefficient equals

$$\frac{1}{24\theta(z; \tau)} + \frac{1}{\theta(z; \tau)} \sum_{n=1}^{\infty} (-1)^n e^{\pi i(n^2+n)\tau} \frac{(1 + e^{2\pi i n \tau})(e^{2\pi i n z} + e^{-2\pi i n z})}{2(1 - e^{2\pi i n \tau})^2}.$$

Therefore in the case  $z = \frac{1+\tau}{2}$  we obtain

$$\begin{aligned}
& -\frac{e^{\frac{\pi i \tau}{4}}}{\theta_1(\tau)} \left( \frac{1}{24} + \sum_{n=1}^{\infty} e^{\pi i n^2 \tau} \frac{(1 + e^{2\pi i n \tau})^2}{2(1 - e^{2\pi i n \tau})^2} \right) \\
&= -\frac{e^{\frac{\pi i \tau}{4}}}{\theta_1(\tau)} \left( -\frac{5}{24} + \frac{\theta_1(\tau)}{4} + 2 \sum_{n=1}^{\infty} \frac{e^{\pi i(n^2+2n)\tau}}{(1 - e^{2\pi i n \tau})^2} \right),
\end{aligned}$$

similarly, in the case  $z = \frac{\tau}{2}$

$$i \frac{e^{\frac{\pi i \tau}{4}}}{\theta_2(\tau)} \left( -\frac{5}{24} + \frac{\theta_2(\tau)}{4} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n e^{\pi i(n^2+2n)\tau}}{(1 - e^{2\pi i n \tau})^2} \right),$$



and finally, in the case  $z = \frac{1}{2}$ :

$$-\frac{1}{\theta_3(\tau)} \left( \frac{1}{24} + \sum_{n=1}^{\infty} e^{\pi i(n^2+n)\tau} \frac{1 + e^{2\pi i n \tau}}{(1 - e^{2\pi i n \tau})^2} \right).$$

For applying Proposition 2.3 we also need to compute the coefficients at  $2\pi i s$  of the following expressions (we omit  $\tau$  from the arguments of  $\theta$ ):

$$\begin{aligned} & \frac{i\eta(\tau)^3 \theta(\frac{1}{2} + s) \theta(\frac{\tau}{2})}{\theta(\frac{1-\tau}{2}) \theta(\frac{1}{2}) \theta(s) \theta(\frac{\tau}{2} + s)}, & \frac{i\eta(\tau)^3 \theta(\frac{1}{2} + s) \theta(\frac{1+\tau}{2})}{\theta(-\frac{\tau}{2}) \theta(\frac{1}{2}) \theta(s) \theta(\frac{1+\tau}{2} + s)}, \\ & \frac{i\eta(\tau)^3 \theta(\frac{\tau}{2} + s) \theta(\frac{1+\tau}{2})}{\theta(-\frac{1}{2}) \theta(\frac{\tau}{2}) \theta(s) \theta(\frac{1+\tau}{2} + s)}, \end{aligned}$$

For this we need to compute the Taylor expansions of  $\theta(s)$ ,  $\theta(\frac{1}{2} + s)$ ,  $\theta(\frac{\tau}{2} + s)$  and  $\theta(\frac{1+\tau}{2} + s)$  up to second term with respect to  $2\pi i s$ . We have (denoting by  $\frac{d}{2\pi i d\tau}$ ):

$$\begin{aligned} \theta(s) &= (2\pi i s) i\eta(\tau)^3 \left( 1 + (2\pi i s)^2 \frac{\eta'(\tau)}{\eta(\tau)} \right) + \dots, \\ \theta(\frac{1}{2} + s) &= -\theta_3(\tau) \left( 1 + (2\pi i s)^2 \frac{\theta_3'(\tau)}{\theta_3(\tau)} \right) + \dots, \\ \theta(\frac{\tau}{2} + s) &= -ie^{-\frac{\pi i \tau}{4}} \theta_2(\tau) \left( 1 - \frac{2\pi i s}{2} + (2\pi i s)^2 \left( \frac{1}{8} + \frac{\theta_2'(\tau)}{\theta_2(\tau)} \right) \right) + \dots, \\ \theta(\frac{1+\tau}{2} + s) &= -e^{-\frac{\pi i \tau}{4}} \theta_1(\tau) \left( 1 - \frac{2\pi i s}{2} + (2\pi i s)^2 \left( \frac{1}{8} + \frac{\theta_1'(\tau)}{\theta_1(\tau)} \right) \right) + \dots. \end{aligned}$$

Thus the coefficients at  $2\pi i s$  of the expressions in question are, correspondingly,

$$\begin{aligned} & -\frac{e^{\frac{\pi i \tau}{4}}}{\theta_1(\tau)} \left( \frac{\theta_3'(\tau)}{\theta_3(\tau)} - \frac{\eta'(\tau)}{\eta(\tau)} + \frac{1}{8} - \frac{\theta_2'(\tau)}{\theta_2(\tau)} \right), \\ & -i \frac{e^{\frac{\pi i \tau}{4}}}{\theta_2(\tau)} \left( \frac{\theta_3'(\tau)}{\theta_3(\tau)} - \frac{\eta'(\tau)}{\eta(\tau)} + \frac{1}{8} - \frac{\theta_1'(\tau)}{\theta_1(\tau)} \right), \\ & \frac{1}{\theta_3(\tau)} \left( \frac{\theta_2'(\tau)}{\theta_2(\tau)} - \frac{\eta'(\tau)}{\eta(\tau)} - \frac{\theta_1'(\tau)}{\theta_1(\tau)} \right). \end{aligned}$$

Putting everything together

$$\begin{aligned} & \left. \frac{d}{2\pi i ds} \right|_{s=0} \mu \left( \frac{1}{2}, \frac{\tau}{2} + s; \tau \right) \\ &= \left. \frac{d}{2\pi i ds} \right|_{s=0} \left( \mu \left( \frac{1-\tau}{2}, s; \tau \right) + \frac{i\eta(\tau)^3 \theta(\frac{1}{2} + s) \theta(\frac{\tau}{2})}{\theta(\frac{1-\tau}{2}) \theta(\frac{1}{2}) \theta(s) \theta(\frac{\tau}{2} + s)} \right) \\ &= -\frac{e^{\frac{\pi i \tau}{4}}}{\theta_1(\tau)} \left( -\frac{1}{12} + \frac{\theta_1(\tau)}{4} + \frac{\theta_3'(\tau)}{\theta_3(\tau)} - \frac{\eta'(\tau)}{\eta(\tau)} - \frac{\theta_2'(\tau)}{\theta_2(\tau)} + 2 \sum_{n=1}^{\infty} \frac{e^{\pi i(n^2+2n)\tau}}{(1 - e^{2\pi i n \tau})^2} \right), \end{aligned}$$

$$\begin{aligned}
& \left. \frac{d}{2\pi i ds} \right|_{s=0} \mu \left( \frac{1}{2}, \frac{1+\tau}{2} + s; \tau \right) \\
&= \left. \frac{d}{2\pi i ds} \right|_{s=0} \left( \mu \left( -\frac{\tau}{2}, s; \tau \right) + \frac{i\eta(\tau)^3 \theta(\frac{1}{2} + s) \theta(\frac{1+\tau}{2})}{\theta(-\frac{\tau}{2}) \theta(\frac{1}{2}) \theta(s) \theta(\frac{1+\tau}{2} + s)} \right) \\
&= -i \frac{e^{\frac{\pi i \tau}{4}}}{\theta_2(\tau)} \left( -\frac{1}{12} + \frac{\theta_2(\tau)}{4} + \frac{\theta'_3(\tau)}{\theta_3(\tau)} - \frac{\eta'(\tau)}{\eta(\tau)} - \frac{\theta'_1(\tau)}{\theta_1(\tau)} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n e^{\pi i(n^2+2n)\tau}}{(1 - e^{2\pi i n \tau})^2} \right),
\end{aligned}$$

$$\begin{aligned}
& \left. \frac{d}{2\pi i ds} \right|_{s=0} \mu \left( \frac{\tau}{2}, \frac{1+\tau}{2} + s; \tau \right) \\
&= \left. \frac{d}{2\pi i ds} \right|_{s=0} \left( \mu \left( -\frac{1}{2}, s; \tau \right) + \frac{i\eta(\tau)^3 \theta(\frac{\tau}{2} + s) \theta(\frac{1+\tau}{2})}{\theta(-\frac{1}{2}) \theta(\frac{\tau}{2}) \theta(s) \theta(\frac{1+\tau}{2} + s)} \right) \\
&= \frac{1}{\theta_3(\tau)} \left( \frac{1}{24} + \frac{\theta'_2(\tau)}{\theta_2(\tau)} - \frac{\eta'(\tau)}{\eta(\tau)} - \frac{\theta'_1(\tau)}{\theta_1(\tau)} + \sum_{n=1}^{\infty} e^{\pi i(n^2+n)\tau} \frac{1 + e^{2\pi i n \tau}}{(1 - e^{2\pi i n \tau})^2} \right).
\end{aligned}$$

The statements we need to prove follow from the following identities:

$$\begin{aligned}
\frac{\theta'_3(\tau)}{\theta_3(\tau)} - \frac{\eta'(\tau)}{\eta(\tau)} - \frac{\theta'_2(\tau)}{\theta_2(\tau)} &= \frac{\theta_1^4(\tau)}{8} - \frac{E_2(\tau)}{24}, \\
\frac{\theta'_3(\tau)}{\theta_3(\tau)} - \frac{\eta'(\tau)}{\eta(\tau)} - \frac{\theta'_1(\tau)}{\theta_1(\tau)} &= \frac{\theta_2^4(\tau)}{8} - \frac{E_2(\tau)}{24}, \\
\frac{\theta'_2(\tau)}{\theta_2(\tau)} - \frac{\eta'(\tau)}{\eta(\tau)} - \frac{\theta'_1(\tau)}{\theta_1(\tau)} &= -\frac{\theta_3^4(\tau)}{8} - \frac{E_2(\tau)}{24},
\end{aligned}$$

□

Propositions 3.4 and 3.5 together give the Fourier expansions of  $\tilde{h}_i$ . Denote

$$\begin{aligned}
h_1(\tau) &= \frac{1}{24\theta_1(\tau)} \left( 2 + E_2(\tau) - 48 \sum_{n=1}^{\infty} \frac{e^{\pi i(n^2+2n)\tau}}{(1 - e^{2\pi i n \tau})^2} \right), \\
h_2(\tau) &= \frac{1}{24\theta_2(\tau)} \left( 2 + E_2(\tau) - 48 \sum_{n=1}^{\infty} \frac{(-1)^n e^{\pi i(n^2+2n)\tau}}{(1 - e^{2\pi i n \tau})^2} \right), \\
h_3(\tau) &= \frac{1}{24\theta_3(\tau)} \left( -1 + E_2(\tau) - 24 \sum_{n=1}^{\infty} e^{\pi i(n^2+n)\tau} \frac{1 + e^{2\pi i n \tau}}{(1 - e^{2\pi i n \tau})^2} \right).
\end{aligned}$$

The series  $h_i$  are holomorphic power series converging on the upper half plane.

Denote

$$\begin{aligned} R_1(\tau) &= \frac{1}{2} \sum_{n \in \mathbb{Z}} |n| \beta(2yn^2) e^{-\pi i n^2 \tau} - \frac{\overline{\theta_1(\tau)}}{2\pi\sqrt{2y}}, \\ R_2(\tau) &= \frac{1}{2} \sum_{n \in \mathbb{Z}} (-1)^n |n| \beta(2yn^2) e^{-\pi i n^2 \tau} - \frac{\overline{\theta_2(\tau)}}{2\pi\sqrt{2y}}, \\ R_3(\tau) &= \frac{1}{2} \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} |\nu| \beta(2y\nu^2) e^{-\pi i \nu^2 \tau} - \frac{\overline{\theta_3(\tau)}}{2\pi\sqrt{2y}}. \end{aligned}$$

**Proposition 3.6.** *For  $i = 1, 2, 3$  we have*

$$\tilde{h}_i(\tau) = h_i(\tau) + R_i(\tau).$$

In his thesis Zweegers also represents  $R$  as a certain integral involving a theta function of weight  $\frac{3}{2}$ . In our case, we also have such a representation but with theta functions of weight  $\frac{1}{2}$ .

**Proposition 3.7.** *For  $i = 1, 2, 3$*

$$R_i(\tau) = \frac{1}{4\pi i} \overline{\int_{\tau}^{i\infty} \frac{\theta_i(z) dz}{(-i(z - \bar{\tau}))^{\frac{3}{2}}}}.$$

*Proof.* Note that the integral on the right converges. We prove the identity termwise using the following formula for a real number  $a$ :

$$\begin{aligned} \frac{1}{2} |a| \beta(2ya^2) e^{-\pi i a^2 \tau} - \frac{\overline{e^{\pi i a^2 \tau}}}{2\pi\sqrt{2y}} &= - \overline{\int_{\tau}^{i\infty} \frac{a^2 i e^{\pi i a^2 z} dz}{2\sqrt{-i(z - \bar{\tau})}} - \frac{e^{\pi i a^2 \tau}}{2\pi\sqrt{2y}}} \\ &= \frac{1}{4\pi i} \overline{\int_{\tau}^{i\infty} \frac{e^{\pi i a^2 z} dz}{(-i(z - \bar{\tau}))^{\frac{3}{2}}}}. \end{aligned}$$

This formula is obtained using the integral representation of  $\beta$

$$\beta(2ya^2) = \int_{2ya^2}^{\infty} e^{-\pi t} \frac{dt}{\sqrt{t}}$$

after the substitution  $t = -i(z - \bar{\tau})a^2$ , and then integration by parts. The case  $a = 0$  should be considered separately.  $\square$

We would like to compute the Fourier coefficients of  $h_i \theta_i$  explicitly. Since we know the Fourier coefficients of  $E_2$ , it remains to consider the following expressions:

**Proposition 3.8.** *We have*

$$\begin{aligned}
2 \sum_{n=1}^{\infty} \frac{e^{\pi i(n^2+2n)\tau}}{(1-e^{2\pi i n\tau})^2} &= \sum_{\substack{m>n>0, \\ m-n \text{ even}}} m e^{\pi i m n \tau} - \sum_{\substack{n>m>0, \\ m-n \text{ even}}} m e^{\pi i m n \tau} \\
2 \sum_{n=1}^{\infty} \frac{(-1)^n e^{\pi i(n^2+2n)\tau}}{(1-e^{2\pi i n\tau})^2} &= \sum_{\substack{m>n>0, \\ m-n \text{ even}}} m(-1)^m e^{\pi i m n \tau} - \sum_{\substack{n>m>0, \\ m-n \text{ even}}} m(-1)^m e^{\pi i m n \tau} \\
\sum_{n=1}^{\infty} e^{\pi i(n^2+n)\tau} \frac{1+e^{2\pi i n\tau}}{(1-e^{2\pi i n\tau})^2} &= \sum_{\substack{m>n>0, \\ m-n \text{ odd}}} m e^{\pi i m n \tau} - \sum_{\substack{n>m>0, \\ m-n \text{ odd}}} m e^{\pi i m n \tau}
\end{aligned}$$

*Proof.* It is clear. □

Looking at the expansions we observe that

$$\begin{aligned}
2 \sum_{n=1}^{\infty} \frac{e^{\pi i(n^2+2n)\tau}}{(1-e^{2\pi i n\tau})^2} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n e^{\pi i(n^2+2n)\tau}}{(1-e^{2\pi i n\tau})^2} \\
= 4 \left( \sum_{m>n>0} m e^{4\pi i m n \tau} - \sum_{n>m>0} m e^{4\pi i m n \tau} \right), \quad (7)
\end{aligned}$$

and

$$\begin{aligned}
2 \sum_{n=1}^{\infty} \frac{e^{\pi i(n^2+2n)\tau}}{(1-e^{2\pi i n\tau})^2} + \sum_{n=1}^{\infty} e^{\pi i(n^2+n)\tau} \frac{1+e^{2\pi i n\tau}}{(1-e^{2\pi i n\tau})^2} \\
= \sum_{m>n>0} m e^{\pi i m n \tau} - \sum_{n>m>0} m e^{\pi i m n \tau} \quad (8)
\end{aligned}$$

Therefore it is not difficult to complete the proof of the following statement:

**Proposition 3.9.** *We have*

$$h_1(\tau)\theta_1(\tau) + h_2(\tau)\theta_2(\tau) - 4 \left( h_1(4\tau)\theta_1(4\tau) + h_3(4\tau)\theta_3(4\tau) \right) = -\frac{\theta_1(2\tau)^4}{4}.$$

Using the corresponding identities between the theta functions, namely

$$\theta_1(\tau) + \theta_2(\tau) = 2\theta_1(4\tau), \quad \theta_1(\tau) + \theta_3(\tau) = \theta_1\left(\frac{\tau}{4}\right),$$

the integral representation of  $R_i$  from Proposition 3.7 and the change of variables

$$\overline{4\pi i R_i(4\tau)} = \int_{4\tau}^{i\infty} \frac{\theta_i(z) dz}{(-i(z-4\bar{\tau}))^{\frac{3}{2}}} = \frac{1}{2} \int_{\tau}^{i\infty} \frac{\theta_i(4z) dz}{(-i(z-\bar{\tau}))^{\frac{3}{2}}}$$

we obtain

$$R_1(\tau)\theta_1(\tau) + R_2(\tau)\theta_2(\tau) - 4(R_1(4\tau)\theta_1(4\tau) + R_3(4\tau)\theta_3(4\tau)) = 0.$$

Therefore we also have the following.

**Proposition 3.10.**

$$\tilde{h}_1(\tau)\theta_1(\tau) + \tilde{h}_2(\tau)\theta_2(\tau) - 4 \left( \tilde{h}_1(4\tau)\theta_1(4\tau) + \tilde{h}_3(4\tau)\theta_3(4\tau) \right) = -\frac{\theta_1(2\tau)^4}{4}.$$

Now we are ready to prove Theorem 1.5. This is done in a series of propositions. Take

$$\tilde{g}(\tau) = -\frac{\tilde{h}_1(2\tau)\theta_1(2\tau) + \tilde{h}_3(2\tau)\theta_3(2\tau)}{2} + \frac{\theta_1(\tau)^4 + \theta_2(\tau)^4}{96},$$

then the following holds.

**Proposition 3.11.** *The function  $\tilde{g}(\tau)$  transforms like a modular form of weight 2:*

$$\tilde{g}(\tau + 1) = \tilde{g}(\tau), \quad \tilde{g}\left(-\frac{1}{\tau}\right) = \tau^2 \tilde{g}(\tau).$$

The function  $\tilde{g}(\tau)$  decomposes as

$$\tilde{g}(\tau) = g(\tau) + r(\tau),$$

where

$$g(\tau) = -\frac{h_1(2\tau)\theta_1(2\tau) + h_3(2\tau)\theta_3(2\tau)}{2} + \frac{\theta_1(\tau)^4 + \theta_2(\tau)^4}{96}$$

and

$$r(\tau) = -\frac{R_1(2\tau)\theta_1(2\tau) + R_3(2\tau)\theta_3(2\tau)}{2}.$$

It is not difficult to compute the Fourier expansion of  $g$ :

**Proposition 3.12.** *We have*

$$g(\tau) = -\frac{E_2(\tau)}{24} - \frac{1}{2} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{nq^{n^2}}{1 - q^n} = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma'(n)q^n,$$

where  $\sigma'(n)$  denotes the sum of positive divisors of  $n$  which are greater than  $\sqrt{n}$ , plus half  $\sqrt{n}$  in the case if  $n$  is a perfect square.

*Proof.* Using the third identity from (3) and (8) we find

$$\begin{aligned} g(\tau) &= \frac{1}{48} \left( -1 - E_2(\tau) + 24 \left( \sum_{m>n>0} me^{2\pi imn\tau} - \sum_{n>m>0} me^{2\pi imn\tau} \right) \right) \\ &= -\frac{1}{24} + \frac{1}{2} \sum_{m>0, n>0} me^{2\pi imn\tau} + \frac{1}{2} \sum_{m>n>0} me^{2\pi imn\tau} - \frac{1}{2} \sum_{n>m>0} me^{2\pi imn\tau} \\ &= -\frac{1}{24} + \sum_{m>0, n>0} me^{2\pi imn\tau} - \sum_{n \geq m>0} me^{2\pi imn\tau} + \frac{1}{2} \sum_{n>0} ne^{2\pi in^2\tau}. \end{aligned}$$

Then the statement follows.  $\square$

Now, by Proposition 3.7,  $r(\tau) = R(\overline{D}(r(\tau))) = R(\overline{D}(\tilde{g}(\tau)))$ , where  $R$  is the operator from the introduction, hence  $g$  is the holomorphic part of  $\tilde{g}$ . Proposition 3.12 gives the Fourier expansion of  $g$ , which, as one can easily verify, coincides with  $\mathcal{J}_{-2}$ . Having the transformation properties of  $\tilde{g}$  proved in Proposition 3.11, it remains to check only the uniqueness statement. This is obvious since there are no holomorphic modular forms of weight 2 for  $\mathrm{SL}(2, \mathbb{Z})$ .

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## References

- [ArBeLi] D. Arcara, A. Bertram, and M. Lieblich, *Bridgeland-stable moduli spaces for K-trivial surfaces*, with an appendix by M. Lieblich, arXiv:0708.2247.
- [BrTo] T. Bridgeland and V. Toledano-Laredo, *Stability conditions and Stoke's phenomena*, arXiv:0801.3974.
- [Br08] T. Bridgeland, *Stability conditions on K3 surfaces*, Duke Math. J. Volume 141, Number 2 (2008), 241–291, doi:10.1215/S0012-7094-08-14122-5, arXiv:math/0307164.
- [Br07] T. Bridgeland, *Stability conditions on triangulated categories*, Ann. of Math. (2), Vol 166, no. 2, 317–346, (2007), arXiv:math/0212237.
- [Br06] T. Bridgeland, *Spaces of stability conditions*, arXiv:math/0611510.
- [BrOn] K. Bringmann and K. Ono, *Dyson's ranks and Maass forms*, to appear in Ann. of Math (2).
- [BrOn07] K. Bringmann and K. Ono, *Lifting cusp forms to Maass forms with an application to partitions*, Proc. Natl. Acad. Sci. USA 104 (2007), no. 10, 3725–3731, doi:10.1073/pnas.0611414104.
- [BrOn06] K. Bringmann and K. Ono, *The  $f(q)$  mock theta function conjecture and partition ranks*, Invent. Math. 165 (2006), no. 2, 243–266, doi:10.1007/s00222-005-0493-5.

- [Do] M. R. Douglas, *D-branes on Calabi-Yau manifolds*, in European Congress of Mathematics, Vol. II (Barcelona, 2000), 449–466, Progr. Math., **202**, Birkhäuser, Basel, 2001, arXiv:math/0009209.
- [Dy] F. Dyson, *A walk through Ramanujan's garden*, Lecture given at the Ramanujan Centenary Conference (Illinois, 1987), in Selected Papers of Freeman Dyson with Commentary, 187–208, AMS, 1996.
- [Fr] R. Friedman, *Algebraic surfaces and holomorphic vector bundles*. Universitext. Springer-Verlag, New York, 1998. x+328 pp.
- [HaNa] G. Harder and M. S. Narasimhan, *On the cohomology groups of moduli spaces of vector bundles on curves.*, Math. Ann. 212 (1974/75), 215–248, doi:10.1007/BF01357141.
- [Jo08] D. Joyce, *Configurations in abelian categories. IV. Invariants and changing stability conditions*, Advances in Mathematics Volume 217, Issue 1, 15 January 2008, 125–204, doi:10.1016/j.aim.2007.06.011, arXiv:math/0410268.
- [Jo07a] D. Joyce, *Motivic invariants of Artin stacks and 'stack functions'*, Quarterly Journal of Mathematics, 2007 58(3), 345–392, doi:10.1093/qmath/ham019, arXiv:math/0410268.
- [Jo07b] D. Joyce, *Holomorphic generating functions for invariants counting coherent sheaves on Calabi-Yau 3-folds*, Geometry & Topology 11 (2007) 667–725, doi:10.2140/gt.2007.11.667, arXiv:hep-th/0607039.
- [KoSo] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*.
- [Ko] M. Kontsevich, *Homological algebra of mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 120–139, Birkhäuser, Basel, 1995.
- [Me] A. Mellit, *Higher Green's functions for modular forms*, Bonn PhD thesis, in preparation.
- [Mu] S. Mukai, *On the moduli space of bundles on K3 surfaces, I*, Vector bundles on algebraic varieties (Bombay, 1984), 341–413, Tata Inst. Fund. Res. Stud. Math., 11, Tata Inst. Fund. Res., Bombay, 1987.
- [NaYo] H. Nakajima and K. Yoshioka, *Instanton counting and Donaldson invariants*. (Japanese) Sūgaku 59 (2007), no. 2, 131–153.
- [Ok07a] S. Okada, Talk at *AMS Summer Research Conference, Derived Categories in Mathematics and Physics*, Snowbird, Utah, July 2007.
- [Ok07b] S. Okada, *Topologies on a triangulated category*, arXiv:math/0701507.

- [Ra00] S. Ramanujan, *Collected Papers of Srinivasa Ramanujan* (Ed. G. H. Hardy, P. V. S. Aiyar, and B. M. Wilson). Providence, RI: Amer. Math. Soc., 2000.
- [Ra88] S. Ramanujan, *The lost notebook and other unpublished papers*. Springer-Verlag, Berlin; Narosa Publishing House, New Delhi, 1988.
- [ST] P. Seidel and R. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108** (2001), no. 1, 37–108, doi:10.1215/S0012-7094-01-10812-0, arXiv:math/000104.
- [To] Y. Toda, *Moduli stacks and invariants of semistable objects on K3 surfaces*, arXiv:math/0703590.
- [Yo] K. Yoshioka, *Irreducibility of moduli spaces of vector bundles on K3 surfaces*, arXiv:math/9907001.
- [Za07a] D. Zagier, *Ramanujan’s mock theta functions and their applications*, Séminaire Bourbaki, 60<sup>ème</sup> année, 2006–2007, n<sup>o</sup> 986.
- [Za07b] D. Zagier, *The dilogarithm functions*, Frontiers in number theory, physics, and geometry. II, 3–65, Springer, Berlin, 2007.
- [Zw] S. P. Zwegers, *Mock theta functions*, Utrecht PhD thesis, 2002, <http://igitur-archive.library.uu.nl/dissertations/2003-0127-094324/inhoud.htm>.